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On dense subsets of the boundary of a Coxeter system

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The purpose of this note is to introduce main results of my recent paper [10] about dense subsets of the boundary of a Coxeter system.

A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S\,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s,t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geq 2$ for each $s,t \in S$ such that $s \neq t$.

The pair (W,S) is called a *Coxeter system*. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group. A subset $T \subset S$ is called a *spherical subset of* S, if the parabolic subgroup W_T is finite. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in S which represents w. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

Let (W,S) be a Coxeter system and let $\mathcal{S}^f$ be the family of spherical subsets of S. We denote $W\mathcal{S}^f$ as the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. The sets $\mathcal{S}^f$ and $W\mathcal{S}^f$ are partially ordered by inclusion. Contractible simplicial complexes $K(W,S)$ and $\Sigma(W,S)$ are defined as the geometric realizations of the partially ordered sets $\mathcal{S}^f$ and $W\mathcal{S}^f$, respectively ([7, §3], [5]). The natural embedding $\mathcal{S}^f \to W\mathcal{S}^f$ defined by $T \mapsto W_T$ induces an embedding $K(W,S) \to \Sigma(W,S)$ which we regard as an inclusion. The group W acts on $\Sigma(W,S)$ via simplicial automorphism. Then $\Sigma(W,S) = WK(W,S)$ ([5], [7]). For each $w \in W$, $wK(W,S)$ is called a *chamber* of $\Sigma(W,S)$. If W is infinite, then $\Sigma(W,S)$ is noncompact. In [12], G. Moussong proved that a natural metric on $\Sigma(W,S)$ satisfies the CAT(0) condition. Hence, if W is infinite, $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([6, §4], [8]). This boundary $\partial\Sigma(W,S)$ is called the *boundary of* (W,S). We note that the natural action of W on $\Sigma(W,S)$ is properly discontinuous and cocompact ([5], [6]), and this action induces an action of W on $\partial\Sigma(W,S)$.

A subset A of a space X is said to be *dense* in X, if $\overline{A} = X$. A subset A of a metric space X is said to be *quasi-dense*, if there exists $N > 0$ such that each point of X is N-close to some point of A.

Let (W,S) be a Coxeter system. Then W has the *word metric* d_ℓ defined by $d_\ell(w,w') = \ell(w^{-1}w')$ for each $w,w' \in W$.

Here we obtained the following theorems in [10].

Theorem 1. *Let (W,S) be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in W with respect to the word metric and $m(s_0,t_0) = \infty$ for some $s_0, t_0 \in S$. Then there exists $\alpha \in \partial\Sigma(W,S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W,S)$.*

Suppose that a group Γ acts properly and cocompactly by isometries on a CAT(0) space X. Every element $\gamma \in \Gamma$ such that the order $o(\gamma) = \infty$ is a hyperbolic transformation of X, i.e., there exists a geodesic axis $c : \mathbb{R} \to X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t+a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^i x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

Theorem 2. *Let (W,S) be a Coxeter system. If the set*

$$\bigcup\{W^{\{s\}} \mid s \in S \text{ such that } m(s,t) = \infty \text{ for some } t \in S\}$$

is quasi-dense in W*, then* $\{w^\infty \mid w \in W$ *such that* $o(w) = \infty\}$ *is dense in* $\partial\Sigma(W,S)$.

Remark. For a negatively curved group G and the boundary ∂G of G,

(1) we can show that $G\alpha$ is dense in ∂G for each $\alpha \in \partial G$ by an easy argument, and

(2) it is known that $\{g^\infty \mid g \in G$ such that $o(g) = \infty\}$ is dense in ∂G ([2]).

Example. Let $S = \{s,t,u\}$ and let

$$W = \langle\, S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1\,\rangle.$$

Then (W,S) is a Coxeter system and $W^{\{s\}}$ is quasi-dense in W. On the other hand, for any $\alpha \in \partial\Sigma(W,S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W,S)$ which is a circle. Thus we can not omit the assumption "$m(s_0,t_0) = \infty$" in Theorem 1.

We showed the following lemma in [10].

Lemma 3. *Let* (W,S) *be a Coxeter system. Suppose that there exist a maximal spherical subset* T *of* S *and* $s_0 \in S$ *such that* $m(s_0,t) \geq 3$ *for each* $t \in T$ *and* $m(s_0,t_0) = \infty$ *for some* $t_0 \in T$*. Then* $W^{\{s_0\}}$ *is quasi-dense in* W.

As an application of Theorems 1 and 2, we can obtain the following corollary from Lemma 3.

Corollary 4. *Let* (W,S) *be a Coxeter system. Suppose that there exist a maximal spherical subset* T *of* S *and an element* $s_0 \in S$ *such that* $m(s_0,t) \geq 3$ *for each* $t \in T$ *and* $m(s_0,t_0) = \infty$ *for some* $t_0 \in T$*. Then*

(1) $W\alpha$ *is dense in* $\partial\Sigma(W,S)$ *for some* $\alpha \in \partial\Sigma(W,S)$*, and*

(2) $\{w^\infty \mid w \in W$ *such that* $o(w) = \infty\}$ *is dense in* $\partial\Sigma(W,S)$.

Example. The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of $\mathbb{Z}^2$, and it satisfies the condition of Corollary 4.

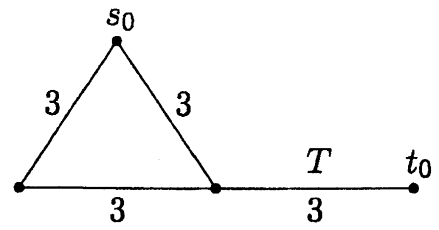

FIGURE 1

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